THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MMAT5000 Analysis I 2015-2016
Problem Set 6: Differentiation

1. Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by

$$f(x) = \begin{cases} x^2 & \text{if } x \in \mathbb{Q}, \\ \\ 0 & \text{otherwise.} \end{cases}$$

Show that f is differentiable at 0, and find $f'(0)$.

Is f differentiable at any point other than $x = 0$? Why?

2. Prove that if $f : \mathbb{R} \to \mathbb{R}$ is an even function which is differentiable, then the derivative f' is an odd function. Also prove that if $g : \mathbb{R} \to \mathbb{R}$ is an odd function which is differentiable, then the derivative g' is an even function.

3. Use the Mean Value Theorem to prove that $|\sin x - \sin y| \leq |x - y|$ for all $x, y \in \mathbb{R}$.

4. Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable in (a, b). Show that if $\lim_{x \to a} f'(x) = A$, then $f'(a)$ exists and equals to A. (Hint: Use the definition of $f'(a)$ and the Mean Value Theorem.)

5. Give an example of a uniformly continuous function on $[0, 1]$ that is differentiable on $(0, 1)$ but whose derivative is not bounded on $(0, 1)$.

6. Let I be an interval. Prove that if f is differentiable on I and if the derivative f' is bounded on I, then f satisfies a Lipschitz condition on I.

7. Suppose that $f, g : \mathbb{R} \to \mathbb{R}$ are two differentiable functions and that

$$\begin{cases} f'(x) = g'(x) \text{ and } g'(x) = -f(x) \text{ for all } x \in \mathbb{R} \\ \\ f(0) = 0 \text{ and } g(0) = 1. \end{cases}$$

Prove that

$$[f(x)]^2 + [g'(x)]^2 = 1 \text{ for all } x \in \mathbb{R}.$$

(Remark: By using the results in ordinary differential equations, we can prove the existence and uniqueness of the functions f and g. Then we can define $\cos x$ and $\sin x$ to be $f(x)$ and $g(x)$ respectively.)

8. A differentiable function $f : I \to \mathbb{R}$ is said to be **uniformly differentiable** on $I := [a, b]$ if for every $\epsilon > 0$ there exist $\delta > 0$ such that if $0 < |x - y| < \delta$ and $x, y \in I$, then

$$\left| \frac{f(x) - f(y)}{x - y} - f'(x) \right| < \epsilon.$$

Show that if f is uniformly differentiable on I, then f' is continuous on I.